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Analytical solutions to steady state unsaturated flow in layered, randomly heterogeneous soils via Kirchhoff transformation

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Abstract

In this study, we derive analytical solutions of the first two moments (mean and variance) of pressure head for one-dimensional steady state unsaturated flow in a randomly heterogeneous layered soil column under random boundary conditions. We first linearize the steady state unsaturated flow equations by Kirchhoff transformation and solve the moments of the transformed variable up to second order in terms of σ_Y and σ_β, the standard deviations of log hydraulic conductivity $Y = \ln(K_s)$ and of the log pore size distribution parameter $\beta = \ln(\alpha)$. In addition, we also give solutions for the mean and variance of the unsaturated hydraulic conductivity. The analytical solutions of moment equations are validated via Monte Carlo simulations.

Keywords: Analytical solutions; Unsaturated flow; Heterogeneity; Uncertainty

1. Introduction

Quantification of uncertainties associated with unsaturated flow in randomly heterogeneous media is challenging. Most relevant studies are numerical, either by Monte Carlo simulations or with numerical moment equation methods [1,3,4,6,8,9,11,16,17,21]. Only a limited number of analytical solutions to the stochastic unsaturated flow problem are available in the literature. Yeh et al. [19] used spectral representations of heterogeneous soil properties to derive solutions of pressure head statistics for gravity-dominated flow (of unit mean gradient). Zhang et al. [22] gave analytical solutions of pressure head variance for gravity-dominated flow with both Gardner–Russo and Brooks–Corey constitutive models. Indelman et al. [7] derived expressions for pressure head moments for one-dimensional steady state unsaturated flow in bounded single-layered heterogeneous formations under deterministic boundary conditions (a constant head at the bottom and constant flux at the top). These expressions contain integrals that have to be evaluated numerically in general. Tartakovsky et al. [15], using the Kirchhoff transformation, solved the mean pressure head and head variance for the one-dimensional unsaturated flow problem up to second order in terms of variability of log saturated hydraulic conductivity. Although their equations were given in a more general form, the analytical solution for the one-dimensional problem is restricted to a special case of a single-layered soil column with a deterministic pore size distribution parameter, under deterministic boundary conditions.

In this paper, we first present analytical solutions (up to second order) for the statistics (mean and variance) of pressure head and unsaturated conductivity for one-dimensional steady state unsaturated flow in a single-layered heterogeneous soil column with random boundary conditions, under the assumptions that the constitutive relationship between pressure head and unsaturated hydraulic conductivity follows the Gardner model and that the pore size distribution parameter α is a random constant in the layer. The solutions are valid for an entire soil column. We then extend our solutions to problems with multiple layers, where the statistics of soil properties in each of these layers may be different. Our solutions are verified using high resolution Monte Carlo simulations.

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2. Mathematical formulation

We start from the steady state equation for flow in a one-dimensional unsaturated heterogeneous single-layered soil column

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(K(z,\psi)\frac{\mathrm{d}h}{\mathrm{d}z}\right) = 0, \quad a \leqslant z \leqslant b, \tag{1}$$

with a constant head boundary at the bottom $z = a$

$$h(a) = H_a, \tag{2}$$

and a constant flux boundary at the top $z = b$

$$K(z,\psi)\frac{\mathrm{d}h}{\mathrm{d}z}\bigg|_{z=b} = -q, \tag{3}$$

where ψ is the pressure head, $h = \psi + z$ is the total head, $H_a = \Psi_a + a$ is the total head at elevation a, K is the unsaturated hydraulic conductivity, q is the flux, and z the vertical coordinate pointing upward. Under this coordinate system, the infiltration rate q is negative. Here we assume that both boundary conditions follow some probability distributions characterized by their means and variances (i.e., the boundary conditions are specified with some uncertainties). To be clear later, specifying boundary conditions with uncertainties allows us to extend our solutions to a soil column with multiple layers.

Integrating (1) in space and using (3) yields a first-order ordinary differential equation for h:

$$K(z,\psi)\frac{\mathrm{d}h}{\mathrm{d}z} = -q, \tag{4}$$

with a boundary condition (2).

To solve (4), it is required to specify some constitutive relationship between K and ψ. No universal models are available for the constitutive relationships. Instead, several empirical models are usually used, including the Gardner–Russo model [5,13], the Brooks–Corey model [2], and the van Genuchten–Mualem model [17]. In most stochastic models of unsaturated flow, the Gardner–Russo model is used due to its simplicity. In this study, we also use the Gardner's model for mathematical convenience:

$$K(z) = K_s(z)\exp[\alpha(z)\psi(z)], \tag{5}$$

where $\alpha(z)$ is the soil parameter related to the pore size distribution. In this study, we treat $K_s(z)$ as a spatially correlated random function following a log–normal distribution, which is consistent with the finding of Russo and Bouton [14] based on field data. For mathematical convenience, we consider the soil parameter α as a random constant, i.e., being a constant in a layer while varying in probability space, with a log–normal distribution. The validation of this assumption is examined numerically in our examples.

Because of randomness in medium properties and boundary conditions, the governing equations (4) and (2) become a set of stochastic differential equations whose solutions are no longer deterministic values but probability distributions or related quantities such as statistical moments of the dependent variables. Our aim is to find the mean head and its associated uncertainty.

Eq. (4) is nonlinear. Upon applying the Kirchhoff transformation $\Phi(z) = \frac{1}{\alpha}\exp[\alpha h(z)]$, (4) becomes a linear ordinary differential equation:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = -\frac{q}{K_s(z)}\mathrm{e}^{\alpha z}, \tag{6}$$

with a boundary condition corresponding to (2)

$$\Phi(a) = \frac{1}{\alpha}\mathrm{e}^{\alpha H_a}. \tag{7}$$

The reason we use the total head h rather than the pressure head ψ in this transformation is that the derived equations will be simpler.

2.1. First moments

Because the variability of Φ depends on the input variabilities, i.e., those of the soil properties (K_s and α) and those of the boundary conditions (H_a and q), one may express Φ as an infinite series in the following form: $\Phi(z) = \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots$, where the order of each term in the series is with respect to σ, which is some combination of variabilities of the input variables. After substituting this expansion and the following formal decompositions into (6) and (7): $H_a = \langle H_a\rangle + H_a'$, $q = \langle q\rangle + q'$, $K_s(z) = \exp[Y(z)] = \exp[\langle Y(z)\rangle + Y'(z)] = K_g(z)\sum_{n=0}^{\infty}[Y']^n/n!$, and $\alpha = \exp(\beta) = \exp(\langle\beta\rangle + \beta') = \alpha_g\sum_{n=0}^{\infty}(\beta')^n/n!$, collecting terms at separate order, and noticing that up to second order in σ_β, the standard deviation of variability $\beta = \ln(\alpha)$,

$$\mathrm{e}^{\alpha z} \approx \mathrm{e}^{\alpha_g z}\left[1 + \alpha_g z\beta' + \frac{1}{2}\alpha_g z(1 + \alpha_g z)\beta'^2\right], \tag{8}$$

and

$$\Phi(a) \approx \frac{\mathrm{e}^{\alpha_g\langle H_a\rangle}}{\alpha_g}\Bigg[1 + (\alpha_g\langle H_a\rangle - 1)\beta' + \alpha_g H_a' + \alpha_g\beta' H_a' + \frac{\alpha_g^2\langle H_a\rangle^2 - \alpha_g\langle H_a\rangle + 1}{2}\beta'^2 + \frac{\alpha_g^2}{2}H_a'^2\Bigg], \tag{9}$$

one obtains the following equations for Φ up to second order:

$$\frac{\mathrm{d}\Phi^{(0)}(z)}{\mathrm{d}z} = -\frac{\langle q\rangle}{K_g(z)}\mathrm{e}^{\alpha_g z}, \tag{10}$$

$$\frac{\mathrm{d}\Phi^{(1)}(z)}{\mathrm{d}z} = -\frac{\mathrm{e}^{\alpha_g z}}{K_g(z)}[\alpha_g\langle q\rangle z\beta' - \langle q\rangle Y'(z) + q'], \tag{11}$$

and

$$\frac{\mathrm{d}\Phi^{(2)}(z)}{\mathrm{d}z} = -\frac{\mathrm{e}^{\alpha_g z}}{K_g(z)}\left[\frac{\langle q\rangle\alpha_g z}{2}(1+\alpha_g z)\beta'^2 - \langle q\rangle\alpha_g z\beta' Y'(z) + \frac{\langle q\rangle}{2}[Y'(z)]^2 + \alpha_g z\beta' q' - q' Y'(z)\right], \tag{12}$$

subject to the boundary conditions

$$\Phi^{(0)}(a) = \frac{\mathrm{e}^{\alpha_g\langle H_a\rangle}}{\alpha_g} \triangleq \Phi_a, \tag{13}$$

$$\Phi^{(1)}(a) = \Phi_a[(\alpha_g\langle H_a\rangle - 1)\beta' + \alpha_g H_a'], \tag{14}$$

and

$$\Phi^{(2)}(a) = \Phi_a\left[\alpha_g\beta' + H_a' + \frac{1}{2}(\alpha_g^2\langle H_a\rangle^2 - \alpha_g\langle H_a\rangle + 1)\beta'^2 + \frac{1}{2}\alpha_g^2 H_a'^2\right], \tag{15}$$

where K_g and α_g are the geometric means of the saturated hydraulic conductivity K_s and the pore-size distribution parameter α, respectively. In the following derivation, both K_g and α_g are considered as constants within each layer. By taking the ensemble mean of these equations and their corresponding boundary conditions, and solving these mean equations, one has

$$\langle\Phi^{(0)}(z)\rangle = \Phi_a - \frac{\langle q\rangle}{\alpha_g K_g}(\mathrm{e}^{\alpha_g z} - \mathrm{e}^{\alpha_g a}), \tag{16}$$

$$\langle\Phi^{(1)}(z)\rangle = 0, \tag{17}$$

and

$$\langle\Phi^{(2)}(z)\rangle = \frac{\Phi_a}{2}\alpha_g^2\sigma_{H_a}^2 + \frac{\Phi_a}{2}(1 - \alpha_g\langle H_a\rangle + \alpha_g^2\langle H_a\rangle^2)\sigma_\beta^2 - \frac{\langle q\rangle\sigma_Y^2}{2\alpha_g K_g}(\mathrm{e}^{\alpha_g z} - \mathrm{e}^{\alpha_g a}) - \frac{\langle q\rangle\sigma_\beta^2}{2\alpha_g K_g} \times [(1 - \alpha_g z + \alpha_g^2 z^2)\mathrm{e}^{\alpha_g z} - (1 - \alpha_g a + \alpha_g^2 a^2)\mathrm{e}^{\alpha_g a}]. \tag{18}$$

It can be shown that $\langle\Phi^{(2)}(z)\rangle \geqslant 0$ for $q \leqslant 0$ (i.e., under infiltration).

2.2. Second moments

Multiplying $\Phi^{(1)}(\xi)$ on (11) and (14) and taking the expectation leads to an equation for covariance of the transformed variable Φ

$$\frac{\mathrm{d}C_\Phi(z,\xi)}{\mathrm{d}z} = -\frac{\mathrm{e}^{\alpha_g z}}{K_g}[\langle q\rangle\alpha_g z\langle\beta'\Phi^{(1)}(\xi)\rangle - \langle q\rangle \times \langle Y'(z)\Phi^{(1)}(\xi)\rangle + \langle q'\Phi^{(1)}(\xi)\rangle], \tag{19}$$

with the boundary condition

$$C_\Phi(z,\xi)|_{z=a} = \Phi_a[(\alpha_g\langle H_a\rangle - 1)\langle\beta'\Phi^{(1)}(\xi)\rangle + \alpha_g \times \langle H_a'\Phi^{(1)}(\xi)\rangle], \tag{20}$$

which involves the cross-covariance functions $\langle\beta'\Phi^{(1)}(\xi)\rangle$, $\langle q'\Phi^{(1)}(\xi)\rangle$, $\langle H_a'\Phi^{(1)}(\xi)\rangle$, and $\langle Y'(z)\Phi^{(1)}(\xi)\rangle$.

By writing (11) and (14) in terms of ξ, and multiplying the derived equations by β', taking the ensemble mean, and assuming that β', Y', and q' are independent, we obtain the equation for the covariance $\langle\beta'\Phi^{(1)}(\xi)\rangle$

$$\frac{\mathrm{d}\langle\beta'\Phi^{(1)}(\xi)\rangle}{\mathrm{d}\xi} = -\frac{\langle q\rangle\alpha_g\sigma_\beta^2}{K_g}\xi\mathrm{e}^{\alpha_g\xi}, \tag{21}$$

subject to the following boundary condition:

$$\langle\beta'\Phi^{(1)}(\xi)\rangle|_{\xi=a} = \Phi_a(\alpha_g\langle H_a\rangle - 1)\sigma_\beta^2. \tag{22}$$

Here we have utilized the fact that β and H_a are uncorrelated, i.e., $\langle\beta' H_a'\rangle \equiv 0$, at the particular boundary conditions in our problem.

Similarly, the equations and their corresponding boundary conditions for covariance $\langle q'\Phi^{(1)}(\xi)\rangle$, $\langle H_a'\Phi^{(1)}(\xi)\rangle$, and $\langle Y'(z)\Phi^{(1)}(\xi)\rangle$ are given as:

$$\frac{\mathrm{d}\langle q'\Phi^{(1)}(\xi)\rangle}{\mathrm{d}\xi} = -\frac{\sigma_q^2}{K_g}\mathrm{e}^{\alpha_g\xi}, \tag{23}$$

$$\langle q'\Phi^{(1)}(\xi)\rangle|_{\xi=a} = \Phi_a\alpha_g\langle q' H_a'\rangle, \tag{24}$$

$$\frac{\mathrm{d}\langle H_a'\Phi^{(1)}(\xi)\rangle}{\mathrm{d}\xi} = -\frac{\langle q' H_a'\rangle}{K_g}\mathrm{e}^{\alpha_g\xi}, \tag{25}$$

$$\langle H_a'\Phi^{(1)}(\xi)\rangle|_{\xi=a} = \Phi_a\alpha_g\sigma_{H_a}^2, \tag{26}$$

and

$$\frac{\mathrm{d}\langle Y'(z)\Phi^{(1)}(\xi)\rangle}{\mathrm{d}\xi} = \frac{\langle q\rangle}{K_g}C_Y(z,\xi)\mathrm{e}^{\alpha_g\xi}, \tag{27}$$

$$\langle Y'(z)\Phi^{(1)}(\xi)\rangle|_{\xi=a} = 0. \tag{28}$$

Eqs. (21)–(26) can be easily solved:

$$\langle\beta'\Phi^{(1)}(\xi)\rangle = \Phi_a(\alpha_g\langle H_a\rangle - 1)\sigma_\beta^2 - \frac{\langle q\rangle\sigma_\beta^2}{\alpha_g K_g} \times [(\alpha_g\xi - 1)\mathrm{e}^{\alpha_g\xi} - (\alpha_g a - 1)\mathrm{e}^{\alpha_g a}], \tag{29}$$

$$\langle q'\Phi^{(1)}(\xi)\rangle = \Phi_a\alpha_g\langle q' H_a'\rangle - \frac{\sigma_q^2}{\alpha_g K_g}(\mathrm{e}^{\alpha_g\xi} - \mathrm{e}^{\alpha_g a}), \tag{30}$$

and

$$\langle H_a'\Phi^{(1)}(\xi)\rangle = \Phi_a\alpha_g\sigma_{H_a}^2 - \frac{\langle q' H_a'\rangle}{\alpha_g K_g}(\mathrm{e}^{\alpha_g\xi} - \mathrm{e}^{\alpha_g a}). \tag{31}$$

Eqs. (27) and (28) involve the covariance function of log hydraulic conductivity, $C_Y(z,\xi)$. For convenience, we assume $C_Y(z,\xi)$ is an exponential function, i.e., $C_Y(z,\xi) = \sigma_Y^2\exp(-|z-\xi|/\lambda)$, where λ is the correlation length of log hydraulic conductivity. Because $C_\Phi(z,\xi)$ is symmetric with respect to z and ξ, to find the variance of Φ, we only need to solve $\langle Y'(z)\Phi^{(1)}(\xi)\rangle$ from (27) and (28) for the case of $z \leqslant \xi$:

$$\langle Y'(z)\Phi^{(1)}(\xi)\rangle = \frac{\langle q\rangle \lambda\sigma_Y^2}{K_g}\left[\frac{2e^{\alpha_g z}}{1-\alpha_g^2\lambda^2} - \frac{e^{\alpha_g a-(z-a)/\lambda}}{\alpha_g\lambda+1} + \frac{e^{\alpha_g \xi-(\xi-z)/\lambda}}{\alpha_g\lambda-1}\right]. \tag{32}$$

By substituting (29)–(32) into (19) and (20), solving for $C_\Phi(z,\xi)$, and setting $\xi = z$, we obtain the variance $\sigma_\Phi^2(z)$

$$\begin{aligned}\sigma_\Phi^2(z) = {} & \Phi_a^2\alpha_g^2\sigma_{H_a}^2 + \Phi_a^2(\alpha_g\langle H_a\rangle - 1)^2\sigma_\beta^2 - \frac{2\Phi_a}{K_g}\langle q' H_a'\rangle \\ & \times [e^{\alpha_g z} - e^{\alpha_g a}] - \frac{2\langle q\rangle \Phi_a\sigma_\beta^2}{\alpha_g K_g}(\alpha_g\langle H_a\rangle - 1) \\ & \times [(\alpha_g z - 1)e^{\alpha_g z} - (\alpha_g a - 1)e^{\alpha_g a}] \\ & + \frac{\langle q\rangle^2\sigma_\beta^2}{\alpha_g^2 K_g^2}[(\alpha_g z - 1)e^{\alpha_g z} - (\alpha_g a - 1)e^{\alpha_g a})]^2 \\ & + \frac{\sigma_q^2}{\alpha_g^2 K_g^2}(e^{\alpha_g z} - e^{\alpha_g a})^2 + \frac{\langle q\rangle^2\lambda\sigma_Y^2 e^{2\alpha_g a}}{\alpha_g K_g^2(1-\alpha_g^2\lambda^2)} \\ & \times [2\alpha_g\lambda e^{(\alpha_g - 1/\lambda)(z-a)} - (1-\alpha_g\lambda)e^{2\alpha_g(z-a)} - (1+\alpha_g\lambda)]\end{aligned} \tag{33}$$

The first four terms on the right hand side of (33) are the contributions of the input variabilities through the lower boundary, while the remaining three terms on the right hand side are the contributions of the respective β, q, and Y variabilities. Note that both terms with σ_q^2 and σ_Y^2 are in the order of $\exp(2\alpha_g z)$ for large values of z, while the term with σ_β^2 is in the order of $(\alpha_g z - 1)^2\exp(2\alpha_g z)$. Because a large σ_Φ^2 corresponds to a large head variance (see next subsection), this explains why as the increase of elevation z, the contribution of σ_β^2 to head variance is much more important than that of σ_Y^2 [9,23].

2.3. Conversion from Φ to h

As the variable Φ is only an intermediate variable, once the first and second moments of transformed variable Φ are solved, we must transform them back to the original variable, the total head h. By writing $h(z) = \sum_{n=1}^{\infty} h^{(n)}(z)$, recalling the expansions for α and Φ, and substituting these into relationship $\alpha h(z) = \ln[\alpha\Phi(z)]$, we have

$$\begin{aligned}\alpha_g(1+\beta'+\cdots)(h^{(0)}+h^{(1)}+\cdots) & = \ln\left[\alpha_g e^{\beta'}\Phi^{(0)}\sum_{n=0}^{\infty}\frac{\Phi^{(n)}}{\Phi^{(0)}}\right] \\ & = \ln(\alpha_g\Phi^{(0)}) + \beta' + \ln\left(1+\sum_{n=0}^{\infty}\frac{\Phi^{(n)}}{\Phi^{(0)}}\right),\end{aligned} \tag{34}$$

where $\Phi^{(0)} = \langle\Phi^{(0)}\rangle$. Expanding the logarithm in the last equation and collecting terms at separate order (up to second order) yields the equations for the total head up to second order

$$\alpha_g h^{(0)}(z) = \ln[\alpha_g\Phi^{(0)}(z)], \tag{35}$$

$$\alpha_g h^{(1)}(z) + \alpha_g\beta' h^{(0)}(z) = \beta' + \Phi^{(1)}(z)/\Phi^{(0)}(z), \tag{36}$$

or

$$h^{(1)}(z) = \left[\frac{1}{\alpha} - h^{(0)}(z)\right]\beta' + \frac{\Phi^{(1)}(z)}{\alpha_g\Phi^{(0)}(z)}, \tag{37}$$

and

$$\alpha_g\left[h^{(2)}(z) + \beta' h^{(1)}(z) + \frac{1}{2}\beta'^2 h^{(0)}(z)\right] = \frac{\Phi^{(2)}(z)}{\Phi^{(0)}(z)} - \frac{1}{2}\left[\frac{\Phi^{(1)}(z)}{\Phi^{(0)}(z)}\right]^2. \tag{38}$$

2.3.1. First moment of head

By taking the mean of (35), (37), and (38) and solve for $\langle h^{(0)}\rangle$, $\langle h^{(1)}\rangle$, and $\langle h^{(2)}\rangle$, we have

$$h^{(0)}(z) = \frac{1}{\alpha_g}\ln[\alpha_g\Phi^{(0)}(z)] = \frac{1}{\alpha_g}\ln\left[e^{\alpha_g\langle H_a\rangle} - \frac{\langle q\rangle}{K_g}(e^{\alpha_g z} - e^{\alpha_g a})\right], \tag{39}$$

$\langle h^{(1)}(z)\rangle \equiv 0$, and

$$\langle h^{(2)}(z)\rangle = -\frac{\sigma_\beta^2}{\alpha_g} + \frac{1}{2}\sigma_\beta^2 h^{(0)}(z) - \frac{\langle\beta'\Phi^{(1)}(z)\rangle}{\alpha_g\langle\Phi^{(0)}(z)\rangle} + \frac{\langle\Phi^{(2)}(z)\rangle}{\alpha_g\langle\Phi^{(0)}(z)\rangle} - \frac{\sigma_\Phi^2(z)}{2\alpha_g\langle\Phi^{(0)}(z)\rangle^2}. \tag{40}$$

For unsaturated flow, up to first-order, (39) implies $0 < \alpha\Phi^{(0)}(z) < \exp(\alpha z)$. This requires that $-\langle q\rangle \leqslant K_g(1 - e^{\alpha_g(\langle H_a\rangle - z)})/(1 - e^{\alpha_g(a-z)})$ for the case of infiltration, or $\langle q\rangle \leqslant K_g e^{\alpha_g\langle H_a\rangle}/(e^{\alpha_g z} - e^{\alpha_g a})$ for evapotranspiration (i.e., $\langle q\rangle > 0$). The latter can be interplated as the maximum water flux at location z and could be used to calculate the maximum evapotranspiration rate at surface.

2.3.2. Second moments of head

The cross-covariance between total head h and other independent variables can be derived from (37), for example

$$\langle q' h^{(1)}(z)\rangle = \frac{\langle q'\Phi^{(1)}(z)\rangle}{\alpha_g\langle\Phi^{(0)}(z)\rangle}, \tag{41}$$

and the variance of the pressure head, which is the same as the variance of the total head, can be derived from (37)

$$\sigma_\psi^2(z) = \sigma_h^2(z) = \frac{\sigma_\beta^2}{\alpha_g^2}(1-\alpha_g h^{(0)}(z))^2 + \frac{2(1-\alpha_g h^{(0)}(z))}{\alpha_g^2\langle \Phi^{(0)}(z)\rangle} \times \langle \beta' \Phi^{(1)}(z)\rangle + \frac{\sigma_\Phi^2(z)}{\alpha_g^2 \langle \Phi^{(0)}(z)\rangle^2}. \quad (42)$$

Note that, in the case of $\sigma_\beta^2 \equiv 0$, (40) and (42) reduce to (46) and (47) of Tartakovsky et al. [15].

2.4. Multi-layer soil column

For a soil column with n layers defined by $z_1 < z_2 < \cdots < z_{n+1}$ and given boundary conditions as infiltration at the top $z = z_{n+1}$ and constant pressure head at the bottom $z = z_1$, solutions can be derived upward sequentially from the bottom to the top layer. An important observation is that, at the given boundary conditions, the variance of the transformed variable Φ (and thus the head variance) in any layer (including the top interface of the layer) is independent of hydraulic properties of all overlying layers. This implies that Φ_k', the perturbation of Φ in the kth layer, is uncorrelated with the hydraulic properties Y and α of the overlying layers.

Now we can outline the solution procedure for the multiple layer systems as follows. Started from the bottom layer ($k = 1$),

1. set $a = z_k$, and $b = z_{k+1}$;
2. solve for first moments of the transformed variable $\Phi^{(0)}$, and $\Phi^{(2)}$ from (16) and (18);
3. solve for variance σ_Φ^2 and cross-covariance $\langle \beta'\Phi^{(1)}\rangle$, $\langle q'\Phi^{(1)}\rangle$, $\langle H_a'\Phi^{(1)}\rangle$, and $\langle Y'\Phi^{(1)}\rangle$ from (29)–(33);
4. compute mean head $h^{(0)}$ and $h^{(2)}$ from (39) and (40);
5. compute head variance σ_ψ^2 using (42);
6. evaluate $\langle q'h^{(1)}\rangle$ at the top boundary of the layer using (41). This value is taken as input to (30) and (31);
7. set $\langle H_a\rangle = h^{(0)}(z_{k+1}) + h^{(2)}(z_{k+1})$ and $\sigma_{H_a}^2 = \sigma_h^2(z_{k+1})$ as the boundary conditions at the bottom of the overlying layer $k+1$ and repeat steps (1)–(6) for each additional overlying layer.

2.5. Unsaturated hydraulic conductivity

From (5) and the expression $\Phi(z) = \frac{1}{\alpha}\exp(\alpha h(z))$, we have

$$K(z) = \alpha K_s \Phi e^{-\alpha z}. \quad (43)$$

Writing $K = K^{(0)} + K^{(1)} + \cdots$, and substituting expansions of α, K_s, and Φ into (43) and separating terms at different order yields

$$K^{(0)}(z) = \alpha_g K_g e^{-\alpha_g z}\Phi^{(0)}(z), \quad (44)$$

$$K^{(1)}(z) = \alpha_g K_g e^{-\alpha_g z}[\Phi^{(1)}(z) + \Phi^{(0)}(z)Y'(z) + (1-\alpha_g z)\Phi^{(0)}(z)\beta'], \quad (45)$$

and

$$K^{(2)}(z) = \alpha_g K_g e^{-\alpha_g z}[\Phi^{(2)}(z) + Y'(z)\Phi^{(1)}(z) + (1-\alpha_g z)\beta'\Phi^{(1)}(z) + \Phi^{(0)}(z)(1-\alpha_g z)\beta' Y'(z) + \Phi^{(0)}(z)\sigma_Y^2/2 + \Phi^{(0)}(z)(1-3\alpha_g z + \alpha_g^2 z^2)\sigma_\beta^2/2]. \quad (46)$$

By taking the expectation of (44)–(46), we have

$$\langle K^{(0)}(z)\rangle = K^{(0)}(z) = \alpha_g K_g e^{-\alpha_g z}\Phi^{(0)}(z) = K_g e^{\alpha_g \Psi_a} - \langle q\rangle(1 - e^{\alpha_g(a-z)}), \quad (47)$$

$\langle K^{(1)}(z)\rangle \equiv 0$, and

$$\langle K^{(2)}(z)\rangle = \alpha_g K_g e^{-\alpha_g z}[\langle \Phi^{(2)}(z)\rangle + \langle Y'(z)\Phi^{(1)}(z)\rangle + (1-\alpha_g z)\langle \beta'\Phi^{(1)}(z)\rangle + \Phi^{(0)}(z)\sigma_Y^2/2 + \Phi^{(0)}(z)(1-3\alpha_g z + \alpha_g^2 z^2)\sigma_\beta^2/2]. \quad (48)$$

The variance of the unsaturated hydraulic conductivity can be derived from (45):

$$\sigma_K^2(z) = \alpha_g^2 K_g^2 e^{-2\alpha_g z}[\sigma_\Phi^2(z) + [\Phi^{(0)}]^2\sigma_Y^2 + (1-\alpha_g z)^2[\Phi^{(0)}]^2\sigma_\beta^2] + 2\Phi^{(0)}\langle Y'(z)\Phi^{(1)}(z)\rangle + 2(1-\alpha_g z)\Phi^{(0)}\langle \beta'\Phi^{(1)}(z)\rangle. \quad (49)$$

3. Illustrative examples

In this section, we demonstrate the accuracy of our second-order analytical solutions of the mean pressure head and the head variance for one-dimensional steady state unsaturated flow in a hypothetical layered soil column, by comparing our results with those from Monte Carlo simulations.

3.1. Base case

In our base case, denoted by Case 1, we consider a one-dimensional heterogeneous soil column with three layers. The length of the soil column is 20 m and the thickness of these layers (from the bottom to the top layer) is 5, 5, and 10 m, respectively. The column is uniformly discretized into 400 line segments (one-dimensional elements) of 0.05 m in length. The origin of the vertical coordinate is set at the bottom of the column. The mean total head is prescribed at the bottom as $\langle H_a\rangle = 0.0$ m (i.e., $\langle \Psi_a\rangle = 0.0$, the water table) and $\sigma_{H_a}^2 = \sigma_{\Psi_a}^2 \equiv 0$, and the mean infiltration rate is given at the top as $\langle q\rangle = -0.002$ m/day with a standard deviation of $\sigma_q = 0.0004$ m/day, i.e., the coefficient of variation $CV_q = 20\%$. The means of the log saturated hydraulic conductivity for three layers are given as $\langle Y\rangle = 0.0$, -2.0, and 2.0 (in the unit of ln[m/day]), respectively, with the coefficient of variation $CV_{K_s} = 100\%$ ($\sigma_Y^2 = 0.693$) for all layers. The correlation length of the log hydraulic conductivity is $\lambda = 1.0$ m for all layers. The statistics of the logarithm of the pore size distribution parameter are given as $\langle\beta\rangle = 0.5$, 1.0, and 0.5 (in the unit of ln[1/m]), respectively, with $CV_\alpha = 10\%$ for all layers.

For the purpose of comparison, we conducted Monte Carlo simulations. For the three layers, we generate three sets of realizations, each of which includes 50,000 one-dimensional unconditional realizations. Each set of these realizations has been tested separately by comparing their sample statistics (the mean, variance, and correlation length) against the input statistics. The comparisons show that the generated random fields reproduce the specified mean and covariance structure very well. Realizations of the log hydraulic conductivity fields for the whole column are then composed by three realizations chosen from each set of realizations.

The steady state unsaturated flow equation, i.e. (1), is solved, using Yeh's algorithm [20], for each realization of the log hydraulic conductivity field together with three independently-generated random numbers representing the logarithm of the pore size distribution parameter for the three layers. If a solution of pressure head contains any positive values (i.e., the column is partially saturated), the realization corresponding to this solution is simply removed. The sample statistics of the flow field, i.e., the mean prediction of head and its associated uncertainty (variance) as well as the mean and variance of unsaturated hydraulic conductivity, are then computed from the rest of realizations. These statistics are considered the "true" solutions that are used to compare against the derived analytical solutions of the moment equations.

Fig. 1a compares the mean pressure head derived from Monte Carlo simulations (the solid line) and zeroth- and second-order analytical solutions (dashed line and dashed-dotted line). It is seen from the figure that while the zeroth-order solution slightly deviates from Monte Carlo results, the second-order solution is almost identical to the latter. A comparison of the standard deviations of pressure head computed from Monte Carlo simulations and analytical solutions is illustrated in Fig. 1b. It shows that the two solutions are very close. Fig. 2 compares the unsaturated hydraulic conductivity statistics resulted from Monte Carlo simulations and analytical solutions to the moment equations. Again, these results are in excellent agreement, though the analytical results are systematically underestimated. The reason for such underestimation is still not clear and

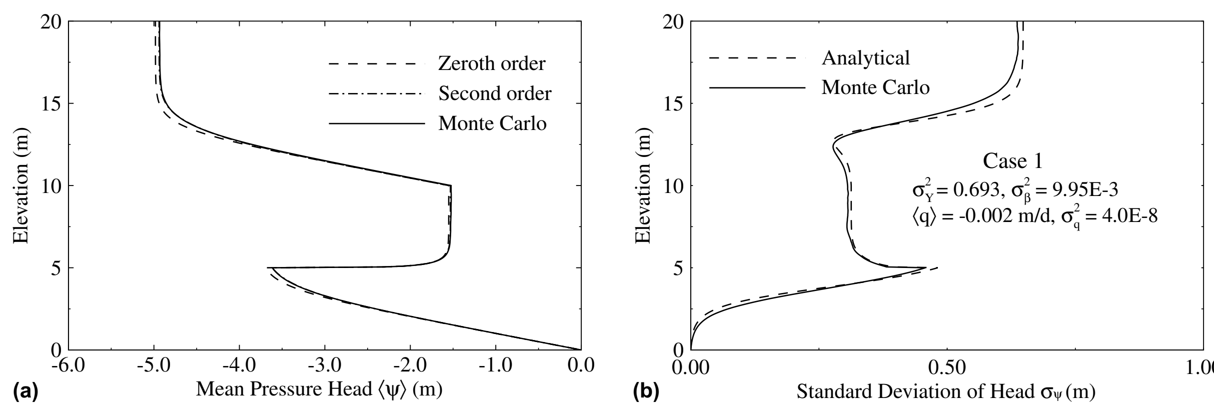

Fig. 1. (a) Mean and (b) standard deviation of pressure head for the base case (Case 1).

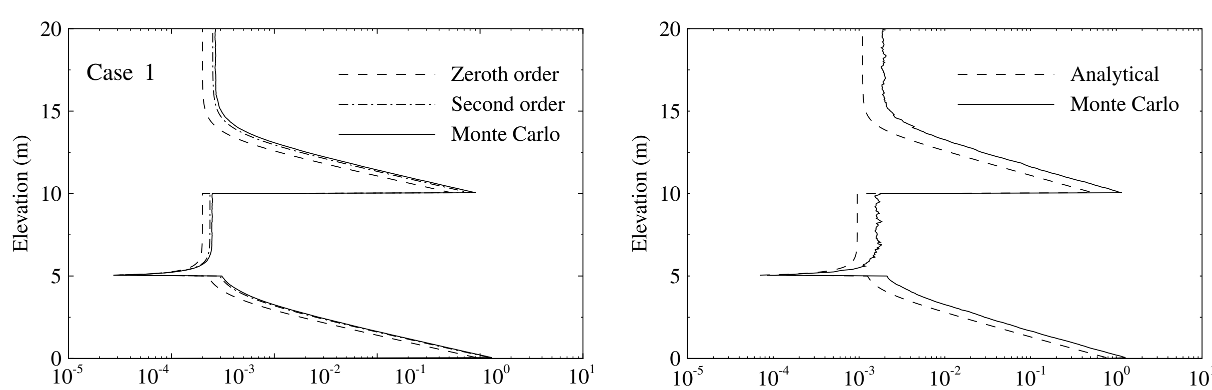

Fig. 2. (a) Mean and (b) standard deviation of unsaturated hydraulic conductivity for the base case (Case 1).

further investigation may be needed. Here we would like to mention that although the variability of Y and β in each layer are not very large, the total variability of either Y or β for the whole column is still relatively large because of the contrast between layers [10].

3.2. Large variability of Y and β

Now we would like to investigate the validity of our solutions at very large variabilities of Y and β. Fig. 3 depicts the comparisons of Monte Carlo results and analytical solutions at $\sigma_Y^2 = 4.0$ (the coefficient of variation $\mathrm{CV}_{K_s} = 732\%$). Note that at such a large variability, the zeroth-order analytical solution of the mean pressure head greatly deviates from the Monte Carlo results. However, after including the second-order corrections the solution is almost identical to the Monte Carlo results. The head variance from our analytical solution is reasonably close to Monte Carlo results. There are two possible reasons that contribute to the discrepancy between the head variances computed from Monte Carlo simulations and analytical solutions. First, the head variance $\sigma_\psi^2[z] = \langle[\psi^{(1)}(z)]^2\rangle$ from the analytical solutions represents the lowest-order approximation of the pressure head variance. Second, due to the large variability on Y, the saturated hydraulic conductivity K_s in some points of realizations is so low that the medium becomes partially saturated and thus these realizations are removed in the Monte Carlo simulations. Overall, 2590 ($= 12.7\%$) out of 20,000 realizations, have been removed for this case.

Fig. 4 compares the Monte Carlo results and analytical solutions for a large β variability $\sigma_\beta^2 = 0.087$ ($\mathrm{CV}_\alpha = 30\%$). A few observations can be made from this figure. First, the analytical solutions are very close to the Monte Carlo results even at such a large variability of β. In addition, if we compare this figure with Fig. 3, we find that the head variance due to $\mathrm{CV}_\alpha = 30\%$ is much larger that that due to $\mathrm{CV}_{K_s} = 732\%$. This finding is consistent with our early conclusion [23] that the contribution of β variability to the head variance is much more important than is the contribution of Y variability.

3.3. Random constant α approximation

Another interesting point we would like to explore is the influence of our assumption (or approximation) of random constant α. We do so with a new set of Monte

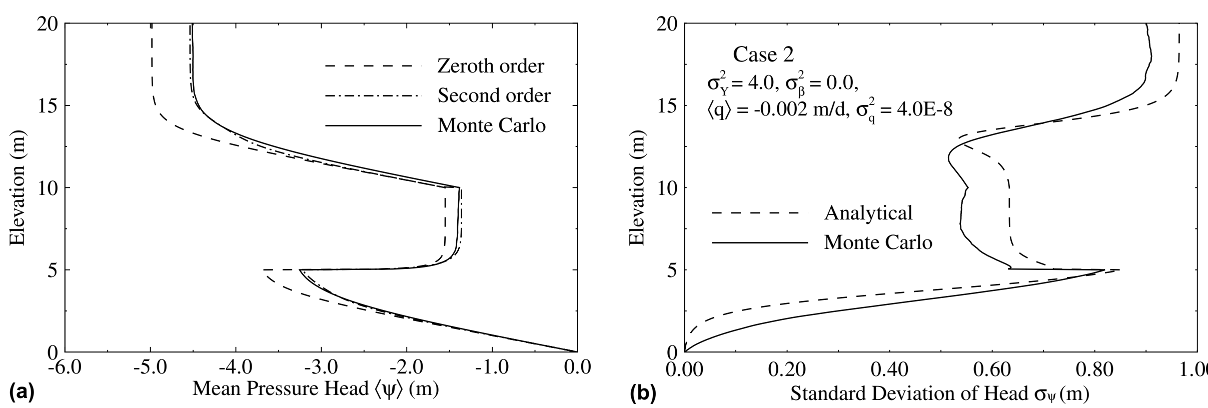

Fig. 3. (a) Mean and (b) standard deviation of pressure head for Case 2: $\sigma_Y^2 = 4.0$ ($\mathrm{CV}_{K_s} = 732\%$) for each layer.

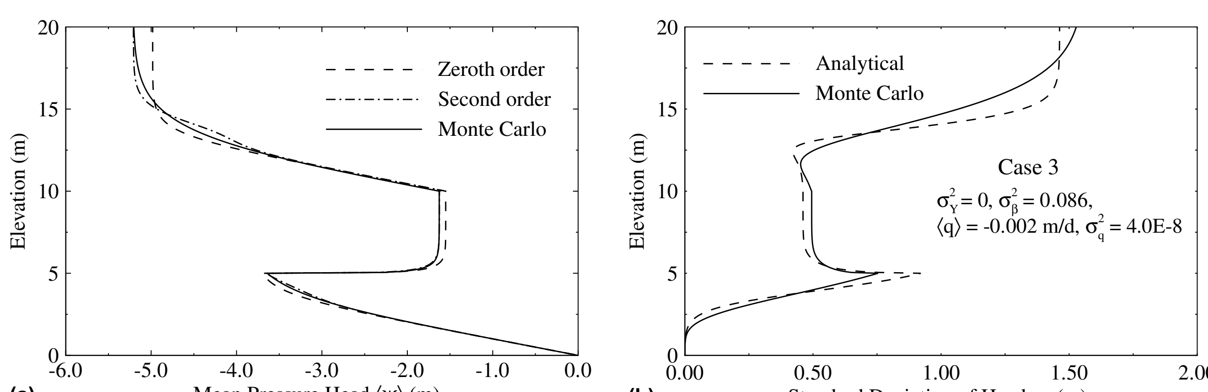

Fig. 4. (a) Mean and (b) standard deviation of pressure head for Case 3: $\sigma_\beta^2 = 0.086$ ($\mathrm{CV}_\alpha = 30\%$) for each layer.

Carlo simulations. In Case 4, instead of generating a random number as the α value for each layer, we now compose the correlated random fields of α for the whole column in the same way as we did for the realizations of Y, already described above.

Fig. 5 compares results from our analytical solutions in which α in each layer is a random constant against those from Monte Carlo simulations where α in each layer is a correlated random function. Comparing it to Fig. 1 shows that our analytical solutions are in excellent agreement with Monte Carlo results in the first and second layer (counting from the bottom) but there is a discrepancy in the top layer, especially for the head variance. A similar pattern is observed by comparing the unsaturated conductivity statistics (not shown here). We suspect that this may be due to the large thickness of this layer (10 correlation length).

In order to check this, we analyzed two more cases. In Case 5, the top layer (10 m) is further divided into two layers with thickness of 5 m each. The properties of the third layer are the same as those of the top layer in Case 4, and the fourth layer are the same as the second layer of Case 4. Again, in Monte Carlo simulations, α in each layer is a correlated random function (correlation length $\lambda = 1.0$), while in our analytical solutions, α is a random constant. The comparison is illustrated in Figs. 6 and 7. It is now seen that the analytical solutions are in excellent agreement with Monte Carlo results. In Case 6, the layer configuration is the same as in Case 4, but now we increase the correlation length of α in the top layer from 1 to 2.5 m, i.e., the top layer is 4 correlation length in thickness (10 m). The results are shown in Fig. 8. Certainly, compared to Fig. 5, the agreement between Monte Carlo results and our analytical solutions has been significantly improved. Fig. 9 shows such comparison for unsaturated hydraulic conductivity.

The results from these two cases imply that when the layer thickness is relatively small (in physical length) or the correlation length of α is relatively large, the correlated random function may be approximated very well by a random constant. These results are consistent with the finding of Yeh et al. [19] and that of Hopmans [6]. In fact, a random constant is a special case of correlated field with a correlation length of infinity.

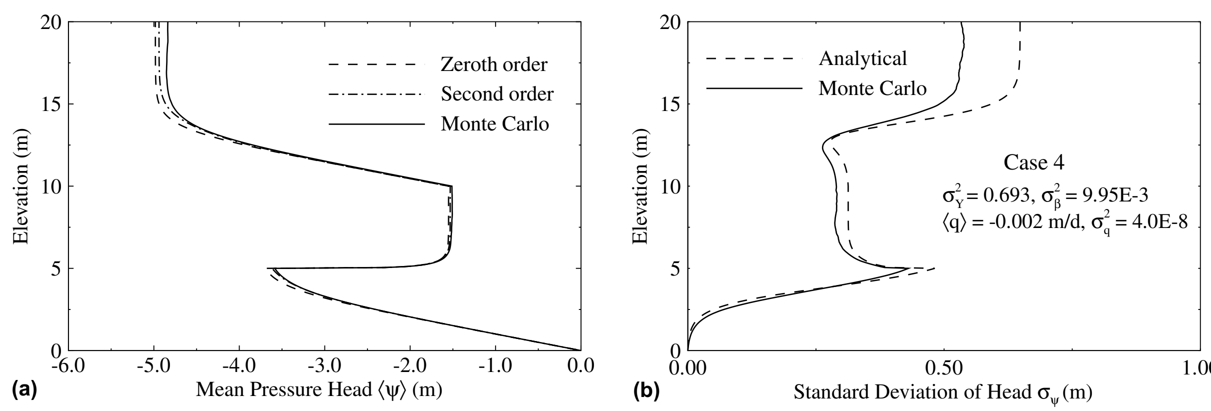

Fig. 5. (a) Mean and (b) standard deviation of pressure head for Case 4. All parameters are similar to the base case, except that β in Monte Carlo simulations is a spatially correlated random function in each layer rather than a random constant.

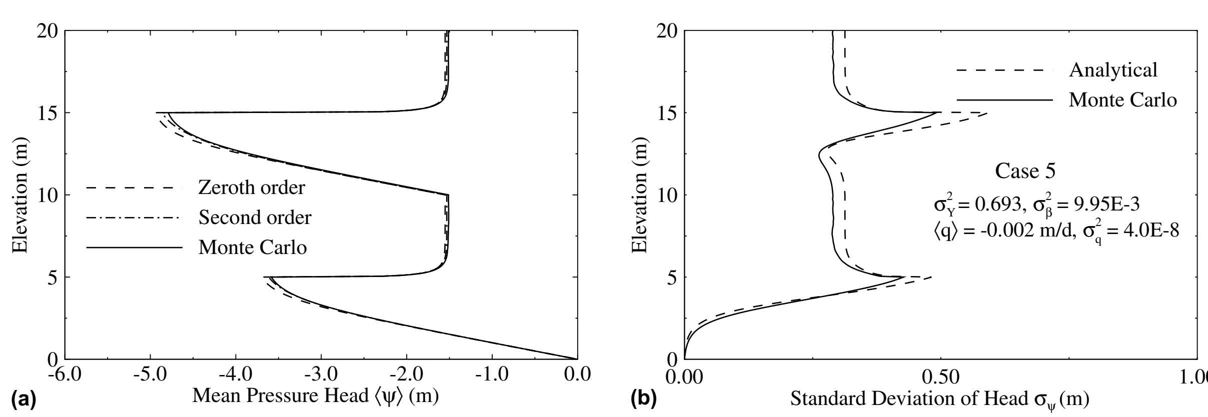

Fig. 6. (a) Mean and (b) standard deviation of pressure head for Case 5.

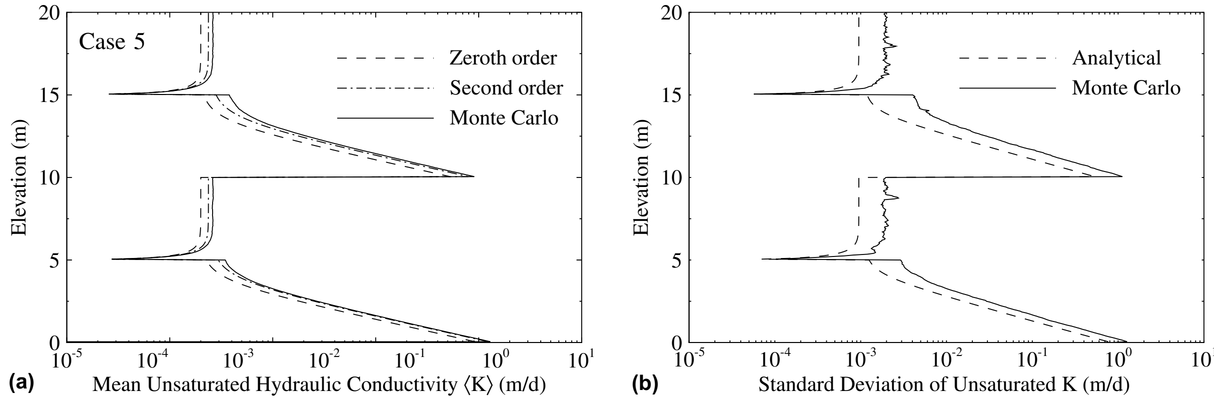

Fig. 7. (a) Mean and (b) standard deviation of unsaturated hydraulic conductivity for Case 5.

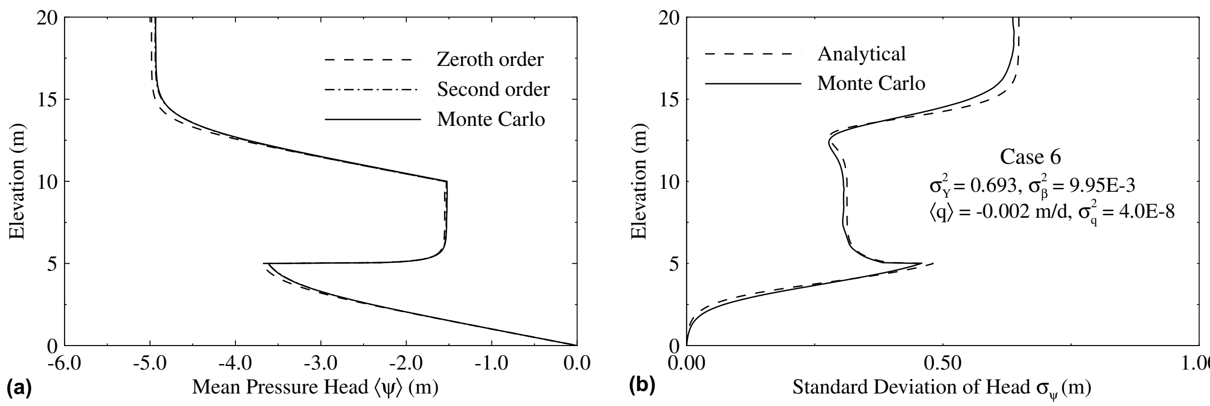

Fig. 8. (a) Mean and (b) standard deviation of pressure head for Case 6.

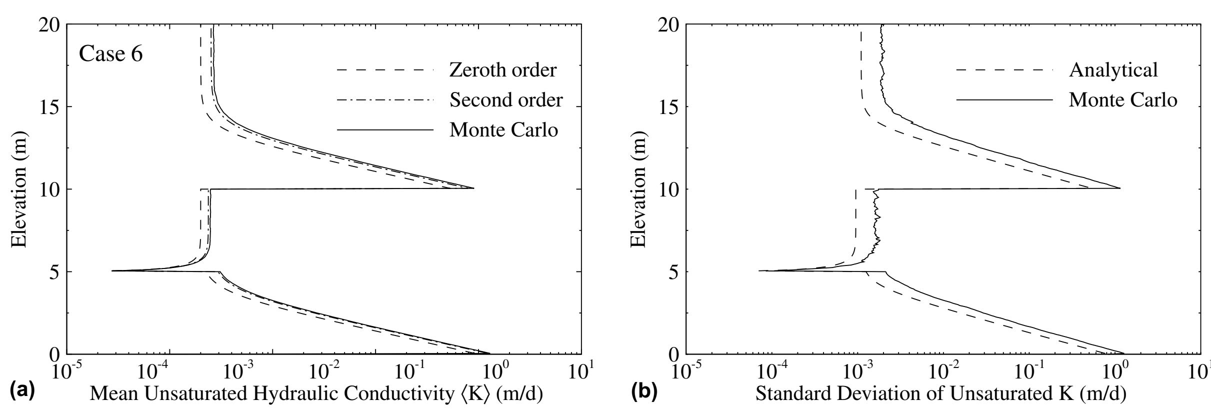

Fig. 9. (a) Mean and (b) standard deviation of unsaturated hydraulic conductivity for Case 6.

4. Summary

We derived analytical solutions of the first two moments (mean and variance) of the pressure head and the unsaturated hydraulic conductivity for one-dimensional steady state unsaturated flow in a randomly heterogeneous layered soil column under random boundary conditions (a prescribed constant head at the bottom and a flux at the top boundary), with an assumption that the constitutive relation between the unsaturated hydraulic conductivity and the pressure head follows Gardner's exponential model. Unlike most of analytical solutions in literature for unsaturated flow in heterogeneous soil column, our solutions are not limited to the gravity-dominated regime but vaild for the entire unsaturated zone. Our solutions are second order in terms of the standard deviations of the log hydraulic conductivity and the pore size distribution parameter. The accuracy of these second order solutions is verified using Monte Carlo simulations. Numerical examples show that these solutions are valid for relatively large variabilities in soil properties.

Our solutions of the first two moments of the pressure head are derived based on the assumption (or approximation) that the pore size distribution parameter α is a random constant in each layer. Numerical examples indicated that such an approximation may be appropriate if the ratio of the correlation length of α in any layer to the layer thickness is relatively large (e.g., 0.25 in Case 6). In the limit that this ratio goes to infinity, the random constant treatment becomes exact.

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